

Solutions of associative Yang-Baxter equation and D -equation in low dimensions and associated Frobenius algebras and Connes cocycles

Mahouton Norbert Hounkonnou* and Gbêvèwou Damien Houndedji†

December 24, 2015

ABSTRACT. This work addresses some relevant characteristics of associative algebras in low dimensions. Especially, given 1 and 2 dimensional associative algebras, we explicitly solve associative Yang-Baxter equations and use skew-symmetric solutions to perform double constructions of Frobenius algebras. Besides, we determine related compatible dendriform algebras and solutions of their D -equations. Finally, using symmetric solutions of the latter equations, we proceed to double constructions of corresponding Connes cocycles.

Keywords. Associative algebra, Frobenius algebra, Connes cocycle, dendriform algebra, D -equation, Yang-Baxter equation.

MSC2010. 16T25, 05C25, 16S99, 16Z05.

1. Introduction

A (symmetric) Frobenius algebra which is an associative algebra with a (symmetric) non-degenerate invariant bilinear form is an important object in both mathematics and mathematical physics. It plays a key role in the study of many topics, such as statistical models over 2-dimensional graphs [7] and topological quantum field theory [19]. On the other hand, a non-degenerate Connes cocycle is an associative algebra with a non-degenerate antisymmetric bilinear form being a cyclic 1-cocycle in the sense of Connes [11]. It corresponds to the original definition of cyclic cohomology by Connes and hence is important in the study of noncommutative geometry.

However, it is not easy to construct Frobenius algebras or non-degenerate Connes cocycles explicitly, that is, both the explicit examples of Frobenius algebras and non-degenerate Connes cocycles are lacked. In [5], some special constructions (namely, double constructions) of both two objects were given in terms of bialgebra structures and certain algebraic equations. In particular, it provides an approach to construct both (symmetric) Frobenius algebras and non-degenerate Connes cocycles from solving certain algebraic equations. Explicitly, a (symmetric) Frobenius algebra can be obtained from an anti-symmetric solution of associative Yang-Baxter equation in an associative algebra, whereas a non-degenerate Connes cocycle can be obtained from a symmetric solution of D -equation in a dendriform algebra which is the underlying algebraic structure of a non-degenerate Connes cocycle. Note that both associative Yang-Baxter equation and dendriform algebras appeared more early in some other fields. For example, the associative Yang-Baxter equation was introduced by Aguiar [1] to study the cases of principal derivations for the infinitesimal bialgebras given by Joni and Rota [18] to provide an algebraic framework for the calculus of divided difference, whereas dendriform algebras were introduced by Loday [20] with motivation from algebraic K-theory and were studied quite extensively with connections to several areas in mathematics and physics, like operads [22], homology [14], [15], arithmetics [21] and quantum field theory [13].

In this paper, under the above framework, we will give the explicit study in low dimensions. The paper is organized as follows. In Section 2, we give some basic notions and results on the double constructions of Frobenius algebras and Connes cocycles. In Section 3, we give the explicit study in dimension 1. Specifically, given 1-dimensional associative algebras, we explicitly solve associative Yang-Baxter equations and use skew-symmetric solutions to perform double constructions of Frobenius algebras. Then, we determine related compatible dendriform algebras and solutions of their D -equations. Finally, using symmetric solutions of the latter equations, we proceed to double constructions of corresponding Connes cocycles. In Section 4, similar calculations are performed in dimension 2. In Section 5, we give some concluding remarks.

2. Preliminaries

In this section, we give a quick overview on main definitions and fundamental results essentially known from [5]. See also [1], [26]- [29], [20] and [16] and the references therein.

DEFINITION 2.1. A bilinear form $\mathcal{B}(\cdot, \cdot)$ on an associative algebra \mathcal{A} is **invariant** if

$$\mathcal{B}(xy, z) = \mathcal{B}(x, yz) \text{ for all } x, y, z \in \mathcal{A}.$$

DEFINITION 2.2. An antisymmetric bilinear form $\omega(\cdot, \cdot)$ on an associative algebra \mathcal{A} is a **cyclic 1-cocycle in the sense of Connes** if

$$\omega(xy, z) + \omega(yz, x) + \omega(zx, y) = 0 \text{ for all } x, y, z \in \mathcal{A}. \quad (2.1)$$

For simplicity, ω is called a **Connes cocycle**.

DEFINITION 2.3. A Frobenius algebra $(\mathcal{A}, \mathcal{B})$ is an associative algebra \mathcal{A} with a non-degenerate invariant bilinear form $\mathcal{B}(\cdot, \cdot)$. It is symmetric if \mathcal{B} is symmetric.

DEFINITION 2.4. We call $(\mathcal{A}, \mathcal{B})$ a double construction of a (symmetric) Frobenius algebra associated to \mathcal{A}_1 and \mathcal{A}_1^* if it satisfies the conditions

- (1) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;
- (2) \mathcal{A}_1 and \mathcal{A}_1^* are associative subalgebras of \mathcal{A} ;
- (3) \mathcal{B} is the natural symmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given by

$$\mathcal{B}(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle \text{ for all } x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ is the natural pair between the vector space \mathcal{A}_1 and its dual space \mathcal{A}_1^* .

We call (\mathcal{A}, ω) a Connes cocycle associated to \mathcal{A}_1 and \mathcal{A}_1^* if it satisfies the conditions (1), (2) and

- (4) ω is the natural antisymmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given by

$$\omega(x + a^*, y + b^*) = -\langle x, b^* \rangle + \langle a^*, y \rangle \text{ for all } x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*, \quad (2.3)$$

and ω is a Connes cocycle on \mathcal{A} .

Let us now give some notations useful in the sequel. Let \mathcal{A} be an associative algebra.

Considering the representations of the left L and right R multiplication operations defined as:

$$\begin{aligned} L : \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ x &\longmapsto L_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & x \cdot y, \end{array} \end{aligned} \quad (2.4)$$

$$\begin{aligned} R : \mathcal{A} &\longrightarrow \mathfrak{gl}(\mathcal{A}) \\ x &\longmapsto R_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & y \cdot x, \end{array} \end{aligned} \quad (2.5)$$

The dual maps L^*, R^* of the linear maps L, R , are defined, respectively, as: $L^*, R^* : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A}^*)$ such that:

$$\begin{array}{rcl}
 L^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}^*) \\
 & & \mathcal{A}^* \longrightarrow \mathcal{A}^* \\
 x \longmapsto L_x^* : & & u^* \longmapsto L_x^* u^* : \mathcal{A} \longrightarrow \mathbb{K} \\
 & & v \longmapsto \langle L_x^* u^*, v \rangle := \langle u^*, L_x v \rangle,
 \end{array} \tag{2.6}$$

$$\begin{array}{rcl}
 R^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}^*) \\
 & & \mathcal{A}^* \longrightarrow \mathcal{A}^* \\
 x \longmapsto R_x^* : & & u^* \longmapsto R_x^* u^* : \mathcal{A} \longrightarrow \mathbb{K} \\
 & & v \longmapsto \langle R_x^* u^*, v \rangle := \langle u^*, R_x v \rangle,
 \end{array} \tag{2.7}$$

for all $x, v \in \mathcal{A}, u^* \in \mathcal{A}^*$, where \mathcal{A}^* is the dual space of \mathcal{A} .

Let $\sigma : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be the exchange operator defined as

$$\sigma(x \otimes y) = y \otimes x,$$

for all $x, y \in \mathcal{A}$.

An associative Yang-Baxter equation (AYBE) in the associative algebra \mathcal{A} is defined by [5]

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \tag{2.8}$$

where $r = \sum_i x_i \otimes y_i \in \mathcal{A} \otimes \mathcal{A}$ and

$$r_{12}r_{13} = \sum_{i,j} x_i x_j \otimes y_i \otimes y_j,$$

$$r_{13}r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i y_j,$$

$$r_{23}r_{12} = \sum_{i,j} x_j \otimes x_i y_j \otimes y_i.$$

DEFINITION 2.5. Let V_1, V_2 be two vector spaces. For a linear map $\phi : V_1 \rightarrow V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \rightarrow V_1^*$ given by

$$\langle v, \phi^*(u^*) \rangle = \langle \phi(v), u^* \rangle$$

for all $v \in V_1, u^* \in V_2^*$.

DEFINITION 2.6. Let \mathcal{A} be an associative algebra. An **antisymmetric infinitesimal bialgebra** structure on \mathcal{A} is a linear map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that

- (1) $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$ defines an associative algebra structure on \mathcal{A}^* ;
- (2) Δ satisfies the following equations:

$$\Delta(x \cdot y) = (id \otimes L(x))\Delta(y) + (R(y) \otimes id)\Delta(x), \tag{2.9}$$

$$\begin{aligned}
 & (L(y) \otimes id - id \otimes R(y))\Delta(x) \\
 & + \sigma[(L(x) \otimes id - id \otimes R(x))\Delta(y)] = 0,
 \end{aligned} \tag{2.10}$$

for all $x, y \in \mathcal{A}$.

We denote this bialgebra structure by (\mathcal{A}, Δ) or $(\mathcal{A}, \mathcal{A}^*)$.

THEOREM 2.7. Let (\mathcal{A}, \cdot) and (\mathcal{A}^*, \circ) be two associative algebras. Then, the following conditions are equivalent:

- (1) there is a double construction of a Frobenius algebra associated with (\mathcal{A}, \cdot) and (\mathcal{A}^*, \circ) ;
- (2) $(\mathcal{A}, \mathcal{A}^*)$ is an antisymmetric infinitesimal bialgebra.

COROLLARY 2.8. *Let \mathcal{A} be an associative algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Suppose that r is antisymmetric. Then the map Δ defined by*

$$\Delta(x) = (\text{id} \otimes L(x) - R(x) \otimes \text{id})r \text{ for all } x \in \mathcal{A} \quad (2.11)$$

induces an associative algebra structure on \mathcal{A}^ such that $(\mathcal{A}, \mathcal{A}^*)$ is an antisymmetric infinitesimal bialgebra if (2.8) is satisfied.*

PROPOSITION 2.9. *Let (\mathcal{A}, \cdot) be an associative algebra and let $r \in \mathcal{A} \otimes \mathcal{A}$ be an antisymmetric solution of the associative Yang-Baxter equation in \mathcal{A} . Then, the corresponding double construction of Frobenius algebra $(\mathcal{AD}(\mathcal{A}), *)$ associated to \mathcal{A} and \mathcal{A}^* is given from the product in \mathcal{A} as follows:*

$$a^* * b^* = a^* \circ b^* = R^*(r(a^*))b^* + L^*(r(b^*))a^*, \quad (2.12)$$

$$x * a^* = x \cdot r(a^*) - r(R^*(x)a^*) + R^*(x)a^*, \quad (2.13)$$

$$a^* * x = r(a^*) \cdot x - r(L^*(x)a^*) + L^*(x)a^*, \quad (2.14)$$

for any $x \in \mathcal{A}, a^, b^* \in \mathcal{A}^*$.*

DEFINITION 2.10. *Let \mathcal{A} be a vector space with two bilinear products denoted by \prec and \succ . Then $(\mathcal{A}, \prec, \succ)$ is called a **dendriform algebra** if, for any $x, y, z \in \mathcal{A}$,*

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ x \succ (y \succ z) &= (x * y) \succ z, \end{aligned}$$

*where $x * y = x \prec y + x \succ y$.*

Let $(\mathcal{A}, \prec, \succ)$ be a dendriform algebra. For any $x \in \mathcal{A}$, let $L_\succ(x), R_\succ(x)$ and $L_\prec(x), R_\prec(x)$ denote the left and right multiplication operators of (\mathcal{A}, \prec) and (\mathcal{A}, \succ) , respectively:

$$L_\succ(x)y = x \succ y, R_\succ(x)y = y \succ x, L_\prec(x)y = x \prec y, R_\prec(x)y = y \prec x,$$

for all $x, y \in \mathcal{A}$. Moreover, let $L_\succ, R_\succ, L_\prec, R_\prec : \mathcal{A} \rightarrow \text{gl}(\mathcal{A})$ be four linear maps with $x \mapsto L_\succ(x), x \mapsto R_\succ(x), x \mapsto L_\prec(x)$, and $x \mapsto R_\prec(x)$, respectively. It is known that the product given by [20]

$$x * y = x \prec y + x \succ y, \text{ for all } x, y \in \mathcal{A}, \quad (2.15)$$

defines an associative algebra. We call $(\mathcal{A}, *)$ the associated associative algebra of $(\mathcal{A}, \prec, \succ)$ and $(\mathcal{A}, \succ, \prec)$ is called a compatible dendriform algebra structure on the associative algebra $(\mathcal{A}, *)$.

THEOREM 2.11. *Let $(\mathcal{A}, *)$ be an associative algebra and let ω be a non-degenerate Connes cocycle. Then, there exists a compatible dendriform algebra structure \succ, \prec on \mathcal{A} given by*

$$\omega(x \succ y, z) = \omega(y, z * x), \quad \omega(x \succ y, z) = \omega(x, y * z) \text{ for all } x, y \in \mathcal{A}. \quad (2.16)$$

COROLLARY 2.12. *Let $(T(\mathcal{A}) = \mathcal{A} \bowtie \mathcal{A}^*, \omega)$ be a double construction of the Connes cocycle. Then, there exists a compatible dendriform algebra structure \succ, \prec on $T(\mathcal{A})$ defined by the equation (2.16). Moreover, \mathcal{A} and \mathcal{A}^* , endowed with this product, are dendriform subalgebras.*

DEFINITION 2.13. *Let \mathcal{A} be a vector space. A **dendriform D-bialgebra** structure on \mathcal{A} is a set of linear maps $(\Delta_\prec, \Delta_\succ, \beta_\prec, \beta_\succ)$ given by $\Delta_\prec, \Delta_\succ : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \beta_\prec, \beta_\succ : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$, such that*

- (a) $(\Delta_\prec^*, \Delta_\succ^*) : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$ defines a dendriform algebra structure $(\succ_{\mathcal{A}^*}, \prec_{\mathcal{A}^*})$ on \mathcal{A}^* ;
- (b) $(\beta_\prec^*, \beta_\succ^*) : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ defines a dendriform algebra structure $(\succ_{\mathcal{A}}, \prec_{\mathcal{A}})$ on \mathcal{A} ;
- (c) the following equations are satisfied

$$\Delta_\prec(x *_{\mathcal{A}} y) = (\text{id} \otimes L_{\prec_{\mathcal{A}}}(x))\Delta_\prec(y) + (R_{\mathcal{A}}(y) \otimes \text{id})\Delta_\prec(y), \quad (2.17)$$

$$\Delta_\succ(x *_{\mathcal{A}} y) = (\text{id} \otimes L_{\prec_{\mathcal{A}}}(x))\Delta_\succ(y) + (R_{\prec_{\mathcal{A}}}(y) \otimes \text{id})\Delta_\succ(y), \quad (2.18)$$

$$\beta_\prec(a^* *_{\mathcal{A}^*} b^*) = (\text{id} \otimes L_{\prec_{\mathcal{A}^*}}(a^*))\beta_\prec(b^*) + (R_{\mathcal{A}^*}(b^*) \otimes \text{id})\beta_\prec(a^*) \quad (2.19)$$

$$\beta_{\succ}(a^* *_{\mathcal{A}^*} b^*) = (\text{id} \otimes L_{\mathcal{A}^*}(a^*))\beta_{\succ}(b^*) + (R_{\prec_{\mathcal{A}^*}}(b^*) \otimes \text{id})\beta_{\succ}(a^*), \quad (2.20)$$

$$(L_{\mathcal{A}}(x) \otimes \text{id} - \text{id} \otimes R_{\prec_{\mathcal{A}}}(x))\Delta_{\prec}(y) + \sigma[(L_{\succ_{\mathcal{A}}}(y) \otimes (-\text{id}) \otimes R_{\mathcal{A}}(y))\Delta_{\prec}(y)] = 0, \quad (2.21)$$

$$(L_{\mathcal{A}^*}(a^*) \otimes \text{id} - \text{id} \otimes R_{\prec_{\mathcal{A}^*}}(a^*))\beta_{\prec}(b^*) + \sigma[(L_{\succ_{\mathcal{A}^*}}(b^*) \otimes (-\text{id}) \otimes R_{\mathcal{A}^*}(b^*))\beta_{\succ}(a^*)] = 0, \quad (2.22)$$

hold for any $x, y \in \mathcal{A}$ and $a^*, b^* \in \mathcal{A}^*$, where $L_{\mathcal{A}} = L_{\succ_{\mathcal{A}}} + L_{\prec_{\mathcal{A}}}$, $R_{\mathcal{A}} = R_{\succ_{\mathcal{A}}} + R_{\prec_{\mathcal{A}}}$, $L_{\mathcal{A}^*} = L_{\succ_{\mathcal{A}^*}} + L_{\prec_{\mathcal{A}^*}}$, $R_{\mathcal{A}^*} = R_{\succ_{\mathcal{A}^*}} + R_{\prec_{\mathcal{A}^*}}$.

We also denote it by $(\mathcal{A}, \mathcal{A}^*, \Delta_{\succ}, \Delta_{\prec}, \beta_{\succ}, \beta_{\prec})$ or simply $(\mathcal{A}, \mathcal{A}^*)$.

THEOREM 2.14. *Let (A, \prec_A, \succ_A) and $(A^*, \prec_{A^*}, \succ_{A^*})$ be two dendriform algebras. Let $(A, *_A)$ and $(A^*, *_A^*)$ be the associated associative algebras, respectively. Then, the following conditions are equivalent:*

- (1) *there is a double construction of the Connes cocycle associated with $(A, *_A)$ and $(A^*, *_A^*)$;*
- (2) *(A, A^*) is a dendriform D -bialgebra.*

COROLLARY 2.15. *Let $(\mathcal{A}, \succ, \prec)$ be a dendriform algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Suppose that r is symmetric and r satisfies the equation*

$$r_{12} * r_{13} = r_{13} \prec r_{23} + r_{23} \succ r_{12}. \quad (2.23)$$

Then, the maps Δ_{\succ} and Δ_{\prec} are defined, respectively, by

$$\begin{aligned} \Delta_{\succ}(x) &= (\text{id} \otimes L(x) - R_{\prec}(x) \otimes \text{id})r_{\succ}, \\ \Delta_{\prec}(x) &= (\text{id} \otimes L_{\prec}(x) - R(x) \otimes \text{id})r_{\prec}, \forall x \in \mathcal{A}, \end{aligned} \quad (2.24)$$

where $r_{\succ} = -r$ and $r_{\prec} = r$ induce a dendriform algebra structure on \mathcal{A}^ such that $(\mathcal{A}, \mathcal{A}^*)$ is a dendriform D -bialgebra. Equation (2.23) is called a **D -equation** in \mathcal{A} .*

PROPOSITION 2.16. *Let $(\mathcal{A}, \succ, \prec)$ be a dendriform algebra and let $r \in \mathcal{A} \otimes \mathcal{A}$ be a symmetric solution of the D -equation in \mathcal{A} . Then, the corresponding double construction of Connes cocycle associated to \mathcal{A} and \mathcal{A}^* is given from the products in \mathcal{A} as follows:*

$$\begin{aligned} a^* \prec b^* &= -R_{\succ}^*(r(a^*))b^* + L^*(r(b^*))a^*, \\ a^* \succ b^* &= R^*(r(a^*))b^* - L_{\prec}^*(r(b^*))a^*, \\ a^* * b^* &= a^* \succ b^* + a^* \prec b^* = R_{\prec}^*(r(a^*))b^* + L_{\succ}^*(r(b^*))a^*, \\ x \succ a^* &= x \succ r(a^*) - r(R^*(x)a^*) + R^*(x)a^*, \\ x \prec a^* &= x \prec r(a^*) + r(R_{\succ}^*(x)a^*) - R_{\succ}^*(x)a^*, \\ x * a^* &= x * r(a^*) - r(R_{\prec}^*(x)a^*) + R_{\prec}^*(x)a^*, \\ a^* \succ x &= r(a^*) \succ x + r(L_{\prec}^*(x)a^*) - L_{\prec}^*(x)a^*, \\ a^* \prec x &= r(a^*) \prec x - r(L_{\succ}^*(x)a^*) + L_{\succ}^*(x)a^*, \\ a^* * x &= r(a^*) * x - r(L_{\succ}^*(x)a^*) + L_{\succ}^*(x)a^* \end{aligned} \quad (2.25)$$

for any $x \in \mathcal{A}, a^, b^* \in \mathcal{A}^*$.*

In the sequel, unless otherwise stated, all the parameters belong to the complex field \mathbb{C} .

3. 1-dimensional associative algebras

In this section, we investigate the solutions of the associative Yang-Baxter equation, dendriform algebras structures and classify the solutions of D -equations in the case of 1-dimensional associative algebras.

3.1. Solutions of the associative Yang-Baxter equation. Let (\mathcal{A}, \cdot) be an associative algebra with a basis $\{e_1\}$ and $r = a_{11}e_1 \otimes e_1 \in \mathcal{A} \otimes \mathcal{A}$. Then the AYBE becomes

$$a_{11}^2(e_1 \cdot e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_1 \otimes e_1 \cdot e_1 - e_1 \otimes e_1 \cdot e_1 \otimes e_1) = 0.$$

PROPOSITION 3.1. *There are only two non-isomorphic 1-dimensional associative algebras. The solutions of the corresponding associative Yang-Baxter equation are given in Table 1.*

Table 1: *Solutions of the 1-dimensional associative Yang-Baxter equation.*

Associative algebra \mathcal{A}	Solutions of the AYBE
$\mathcal{A}_1 : e_1 \cdot e_1 = 0$	$r = a_{11}e_1 \otimes e_1$
$\mathcal{A}_2 : e_1 \cdot e_1 = e_1$	$r = 0$

3.1.1. *Antisymmetric solutions and Frobenius algebra structures.* Using the Proposition 2.9, we obtain the results presented in Table 2.

Table 2: Antisymmetric solutions and Frobenius algebra structures of 1-dimensional associative algebras.

Associative algebra \mathcal{A}	Antisymmetric solutions	Frobenius algebra structures over $\mathcal{A} \oplus \mathcal{A}^*$
\mathcal{A}_1	$r = 0$	$e_1 * e_1 = e_1 * e_1^* = e_1^* * e_1 = 0$
\mathcal{A}_2	$r = 0$	$e_1 * e_1 = e_1; e_1 * e_1^* = e_1^*; e_1^* * e_1 = e_1^*$

3.2. Dendriform algebra structures and classification of solutions of the D -equation.

PROPOSITION 3.2. *The compatible dendriform algebra structures in 1-dimensional associative algebras and related solutions of D -equations are given in Table 3.*

Table 3: *The 1-dimensional dendriform algebras and classification of solutions of D -equations.*

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
$\mathcal{A}_1 : e_1 \cdot e_1 = 0$	$D_1^1 : e_1 \prec e_1 = e_1 \succ e_1 = 0$	$r = a_{11}e_1 \otimes e_1$
$\mathcal{A}_2 : e_1 \cdot e_1 = e_1$	$D_1^2 : e_1 \succ e_1 = \lambda e_1, e_1 \prec e_1 = (1 - \lambda)e_1; \lambda = 0, 1$	$r = a_{11}e_1 \otimes e_1$

Proof Let us consider the associative algebra \mathcal{A}_2 . We set $e_1 \succ e_1 = ae_1; e_1 \prec e_1 = be_1$. Since $e_1 \cdot e_1 = e_1$, then $a + b = 1$. Moreover, we have the equations

$$\begin{aligned} (e_1 \prec e_1) \prec e_1 &= e_1 \prec (e_1 \cdot e_1), \\ (e_1 \succ e_1) \prec e_1 &= e_1 \succ (e_1 \prec e_1), \\ e_1 \succ (e_1 \succ e_1) &= (e_1 \cdot e_1) \succ e_1 \end{aligned}$$

which give $ab = 0; ab = ba$ and $a(a - 1) = 0$, respectively. Then, we obtain $a = 0$ and $b = 1$ or $a = 1$ and $b = 0$ yielding the compatible dendriform algebra structures on \mathcal{A}_2 . The solutions of D -equations in these dendriform algebras are given by direct computation. \square

3.2.1. *Symmetric solutions and Connes cocycles structures.* Using the Proposition 2.16, we obtain the following results in Table 4, where $D_i^j, i, j \in \mathbb{N}^*$, means the i -th dendriform class associated with the j -th class of associative algebra.

Table 4: Symmetric solutions and Connes cocycles of 1-dimensional associative algebras.

Dendriform algebra	Symmetric solutions	Connes cocycles over $\mathcal{A} \oplus \mathcal{A}^*$
D_1^1	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1 * e_1^* = e_1^* * e_1 = e_1^* * e_1^* = 0$
D_1^2	$a_{11}e_1 \otimes e_1$	$e_1^* * e_1^* = -a_{11}e_1^*, e_1^* * e_1 = a_{11}(\lambda - 1)e_1 + \lambda e_1^*, e_1 * e_1 = e_1, e_1 * e_1^* = -a_{11}\lambda e_1 + (1 - \lambda)e_1^*; \lambda = 0, 1$

4. 2-dimensional associative algebras

Using a similar approach as in the previous section, we now consider 2-dimensional associative algebras. Let (\mathcal{A}, \cdot) be an associative algebra with a basis $\{e_1, e_2\}$ and $r = \sum_{i,j=1}^2 a_{ij}e_i \otimes e_j \in \mathcal{A} \otimes \mathcal{A}$. Then, the AYBE (2.8), i.e. $r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$, is satisfied for

$$r_{12} = \sum_{i,j=1}^2 a_{ij}e_i \otimes e_j \otimes 1,$$

$$r_{13} = \sum_{i,j=1}^2 a_{ij}e_i \otimes 1 \otimes e_j,$$

$$r_{23} = \sum_{i,j=1}^2 a_{ij}1 \otimes e_i \otimes e_j.$$

4.1. Solutions of the associative Yang-Baxter equation. The classification of 2-dimensional complex pre-Lie algebras, including the classification of 2-dimensional complex associative algebras, was performed in [9]. Then, the 2-dimensional complex associative algebras can be split into 7 classes [4].

PROPOSITION 4.1. *The solutions of the associative Yang-Baxter equation (2.8) in 2-dimensional associative algebras are given in Table 5.*

Table 5: *Solutions of the 2-dimensional associative Yang-Baxter equation.*

Associative algebra \mathcal{A}	Solutions of the AYBE
$\mathcal{A}_1 : e_1 \cdot e_1 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$
$\mathcal{A}_2 : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{12} = -a_{21} \neq 0$
$\mathcal{A}_3 : e_1 \cdot e_1 = e_1, e_2 \cdot e_1 = e_2$	$\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{12} = -a_{21} \neq 0$
$\mathcal{A}_4 : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2,$ $e_2 \cdot e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} a_{21} \neq 0;$ $\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{22} \neq 0$
$\mathcal{A}_5 : e_i \cdot e_j = 0; i, j = 1, 2$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
$\mathcal{A}_6 : e_2 \cdot e_2 = e_2$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$
$\mathcal{A}_7 : e_1 \cdot e_1 = e_1; e_2 \cdot e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Proof We set $r = \sum_{i,j} a_{ij} e_i \otimes e_j \in \mathcal{A} \otimes \mathcal{A}$, where $i, j = 1, 2$ and $a_{ij} \in \mathbb{C}$. Then, by direct computation of (2.8), we get the results. \square

4.1.1. *Antisymmetric solutions and Frobenius algebra structures.* Using the Proposition 2.9, we get the results of Table 6.

Table 6: Antisymmetric solutions and Frobenius algebra structures of 2-dimensional associative algebras.

Associative algebra \mathcal{A}	Antisymmetric solutions	Frobenius algebra structures over $\mathcal{A} \oplus \mathcal{A}^*$
\mathcal{A}_1	$r = 0$	$e_1 * e_1 = e_2; e_1 * e_2^* = e_2^*; e_2^* * e_1 = e_1^*$
\mathcal{A}_2	$a_{12} e_1 \otimes e_2$ $+ a_{21} e_2 \otimes e_1;$ $a_{12} = -a_{21} \neq 0$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_2^* * e_1 = e_2^*$ $e_2^* * e_2^* = -a_{12} e_2^*; e_2^* * e_1^* = a_{21} e_1^*; e_1 * e_2^* = a_{21} e_1$ $e_2 * e_2^* = -a_{12} e_2 + e_1^*; e_2^* * e_2 = -a_{12} e_2; e_1^* * e_1 = a_{21} e_2 + e_1^*$
\mathcal{A}_3	$a_{12} e_1 \otimes e_2$ $+ a_{21} e_2 \otimes e_1;$ $a_{12} = -a_{21} \neq 0$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1 * e_2^* = e_2^*; e_1^* * e_1 = e_1^*$ $e_2^* * e_2^* = a_{21} e_2^*; e_1^* * e_2^* = a_{21} e_1^*; e_1 * e_1^* = -a_{12} e_2 + e_1^*$ $e_2 * e_2^* = a_{21} e_2; e_2^* * e_1 = -a_{12} e_1; e_2^* * e_2 = a_{21} e_2 + e_1^*$
\mathcal{A}_4	$r = 0$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2 * e_1 = e_2; e_1^* * e_1 = e_1^*$ $e_1 * e_1^* = e_1^*; e_2^* * e_1 = e_2^*; e_2^* * e_2 = e_1^*$
\mathcal{A}_5	$a_{12} e_1 \otimes e_2$ $+ a_{21} e_2 \otimes e_1;$ $a_{12} = -a_{21} \neq 0$	$e_i * e_j = e_i^* * e_j^* = e_i * e_j^* = 0$ $e_i^* * e_j = 0; i, j = 1, 2$
\mathcal{A}_6	$r = 0$	$e_2 * e_2 = e_2; e_2 * e_2^* = e_2^*; e_2^* * e_2 = e_2^*$
\mathcal{A}_7	$r = 0$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_2 * e_2^* = e_2^*$ $e_1^* * e_1 = e_1^*; e_2^* * e_2 = e_2^*; e_1 * e_1^* = e_1^*$

4.2. Dendriform algebra structures and classification of solutions of the D -equation. The classification of 2-dimensional complex dendriform algebras was firstly studied in [28], but

unfortunately with some mistakes. In fact, there exists a natural anti-isomorphism between dendriform algebras

$$F(x \succ_1 y) = F(y) \prec_2 F(x), F(x \prec_1 y) = F(y) \succ_2 F(x).$$

So, the dendriform algebras appear in terms of pairs. For example, consider the following dendriform algebras given by:

$$\begin{aligned} D_1 : e_1 \prec_1 e_1 &= e_1, e_1 \prec_1 e_2 = e_2; \\ D_2 : e_1 \succ_2 e_1 &= e_1, e_2 \succ_2 e_1 = e_2; \end{aligned}$$

and the map F defined by $F(e_1) = e_1, F(e_2) = e_2$. We have $F(e_1 \prec_1 e_1) = e_1 = F(e_1) \succ_2 F(e_1)$; $F(e_1 \prec_1 e_2) = e_2 = F(e_2) \succ_2 F(e_1)$. Therefore, there exists an anti-isomorphism between the dendriform algebras D_1 and D_2 . By this property, one can classify dendriform algebras. For that, let $e_i, e_j \in \mathcal{A}; i, j = 1, 2$. Then,

$$e_i \cdot e_j = e_i \prec e_j + e_i \succ e_j, \text{ where } e_i \succ e_j = \sum_{k=1}^2 a_{ij}^k e_k, e_i \prec e_j = \sum_{k=1}^2 b_{ij}^k e_k. \quad (4.1)$$

Computing the equation (4.1) with the condition $e_i \cdot e_j = \sum_{k=1}^2 (a_{ij}^k + b_{ij}^k) e_k$, we get the compatible dendriform algebra structures on the associative algebra. The solutions of D -equations in these dendriform algebras are then determined by direct computation. There results the following:

PROPOSITION 4.2. *The compatible dendriform algebra structures in 2-dimensional associative algebras and the solutions of related D -equations are given in Table 7.*

Table 7: 2-dimensional dendriform algebras and classification of solutions of the D -equation.

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
$\mathcal{A}_1:$ $e_1 \cdot e_1 = e_2$	$D_{1,\lambda}^1: e_1 \prec e_1 = \lambda e_2$ $e_1 \succ e_1 = (1 - \lambda)e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix},$ $\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} \lambda = 0, a_{21} \neq 0$ $\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{12} \neq a_{21}$ $\lambda = \frac{a_{12}^2 - a_{12}a_{21}}{a_{21}^2 - a_{12}a_{21}}$
$\mathcal{A}_2:$ $e_1 \cdot e_1 = e_1$ $e_1 \cdot e_2 = e_2$	$D_1^2: e_1 \succ e_2 = e_2$ $e_1 \succ e_1 = e_1$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}, a_{22} \neq 0$ $\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} a_{21}, a_{22} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{12} \neq 0$ $a_{21} = \frac{a_{11}a_{22}}{a_{12}}$
	$D_2^2: e_1 \succ e_2 = e_2$ $e_1 \prec e_1 = e_1$	$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$ $\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{21} = a_{12} \neq 0$
	$D_3^2: e_1 \prec e_1 = e_1$ $e_1 \prec e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{12} = a_{21}, a_{11} \neq 0$ $a_{22} = \frac{a_{12}^2}{a_{11}}$

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
	$D_4^2: e_2 \prec e_1 = e_2$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}$
	$e_1 \succ e_2 = e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0$
	$e_1 \prec e_1 = e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{21} = a_{11}; a_{12} = a_{22} \neq 0$
	$e_2 \succ e_1 = -e_2$	
	$D_5^2: e_1 \succ e_2 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$
	$e_1 \succ e_1 = -e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{12} = a_{21} \neq 0$
	$e_1 \prec e_1 = e_1 + e_2$	
$\mathcal{A}_3:$	$D_1^3: e_1 \succ e_1 = e_1$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}$
	$e_2 \prec e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{12} \neq 0$
		$\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{21} = a_{12}, a_{22} \neq 0$
	$D_2^3: e_1 \prec e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$
	$e_2 \prec e_1 = e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{21} = a_{12}; a_{22} = \frac{a_{12}^2}{a_{11}}, a_{11} \neq 0$
	$D_3^3: e_1 \succ e_1 = e_1 - e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$
	$e_1 \prec e_1 = e_2$	$a_{22} = -a_{21} \neq 0$
	$e_2 \prec e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{12} = a_{21} \neq 0$
		$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{22} = -a_{21},$
		$a_{12} = -a_{11}$
	$D_4^3: e_1 \succ e_2 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$
	$e_1 \succ e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$
	$e_1 \prec e_2 = -e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{11} \neq 0; a_{22} = \frac{a_{21}a_{12}}{a_{11}}$
	$e_2 \prec e_1 = e_2$	
$\mathcal{A}_4:$	$D_1^4: e_1 \succ e_2 = e_2$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix};$
	$e_1 \cdot e_1 = e_1$	$\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0;$
	$e_1 \cdot e_2 = e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$
	$e_2 \cdot e_1 = e_2$	$a_{11}, a_{12} \neq 0$
	$D_2^4: e_1 \succ e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$
	$e_1 \succ e_1 = e_1$	$\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}, a_{12} \neq 0$
	$e_2 \succ e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{21} = a_{12} \neq 0$
		$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} a_{11} \neq 0$

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
	$D_3^4: e_1 \succ e_2 = e_2$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$
	$e_1 \succ e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}, a_{21} \neq 0$
	$e_2 \prec e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0$
	$D_4^4: e_1 \succ e_2 = e_2$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix};$
	$e_1 \succ e_1 = -e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0$
	$e_2 \succ e_1 = e_2$	
	$e_1 \prec e_1 = e_1 + e_2$	
$\mathcal{A}_5:$	$D_1^5: e_2 \succ e_2 = \beta e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \beta = 0; \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \beta = 1$
$e_i \cdot e_j = 0;$	$e_2 \prec e_2 = -\beta e_1$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} a_{21} \neq 0, \beta = 1$
$i, j = 1, 2$	$\beta = 0, 1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{22} \neq 0; \beta = 0$
$\mathcal{A}_6: e_2 \cdot e_2 = e_2$	$D_1^6: e_2 \succ e_2 = e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} a_{22} \neq 0$
	$D_2^6: e_2 \succ e_2 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$
	$e_2 \succ e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{22} \neq 0$
	$e_2 \prec e_1 = -e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{11} \neq 0; a_{21} = a_{12}$
	$D_3^6: e_2 \prec e_2 = e_2$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$
	$e_1 \prec e_2 = e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{12} \neq 0$
	$e_1 \succ e_2 = -e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{22} \neq 0$
	$D_4^6: e_1 \succ e_1 = e_2$	$\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}$
	$e_2 \succ e_2 = e_2$	
	$e_1 \prec e_1 = -e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}, a_{22} \neq 0$
$\mathcal{A}_7:$	$D_1^7: e_1 \prec e_2 = e_1$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} a_{21} \neq 0$
$e_1 \cdot e_1 = e_1$	$e_2 \prec e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{22} \neq 0$
	$e_1 \succ e_1 = e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{21} \neq a_{12}$
$e_2 \cdot e_2 = e_2$	$e_1 \succ e_2 = -e_1$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{21} = a_{12} \neq 0, a_{12} = a_{22}$
		$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} a_{12} = a_{22} \neq 0$

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
D_2^7 : $e_1 \succ e_1 = e_1$ $e_2 \succ e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{12} = a_{22} \neq 0$ $\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} a_{11} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{11} = a_{21} \neq 0, a_{22} = a_{12}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$
D_3^7 : $e_2 \prec e_2 = e_2$ $e_1 \succ e_1 = e_1$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}, a_{21} = a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} a_{11} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} a_{11} = a_{12} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{11} \neq 0, a_{12} \neq 0$ $\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{11} \neq 0$
D_4^7 : $e_1 \prec e_2 = -e_2$ $e_2 \prec e_2 = e_2$ $e_1 \succ e_2 = e_2$ $e_1 \succ e_1 = e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} a_{11} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} a_{11} = a_{12} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{11} \neq 0, a_{12} \neq 0$ $\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} a_{21} = a_{11} \neq 0$	$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \lambda = 0; \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} \lambda \neq 0$ $\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \lambda, a_{22} \neq 0$ $\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}, \lambda = -1, a_{21} = a_{11} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} a_{21} = a_{11} \neq 0, \lambda = -1, a_{12} \neq 0$ $\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} a_{22} = a_{12} \neq 0$
$D_{5,\lambda}^7$: $e_1 \prec e_1 = e_1$ $e_2 \prec e_1 = -\lambda e_2$ $e_2 \succ e_1 = \lambda e_2$ $e_2 \succ e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} a_{22} = a_{21} \neq 0$ $\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} a_{22} = a_{12} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{11} \neq 0; a_{22} = a_{21} = a_{12}$ $\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{22} = a_{12} = a_{21} \neq 0$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$
D_6^7 : $e_1 \prec e_1 = e_1$ $e_2 \prec e_1 = -e_1$ $e_2 \succ e_1 = e_1$ $e_2 \succ e_2 = e_2$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} a_{22} = a_{21} \neq 0$ $\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix} a_{22} = a_{12} \neq 0$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{11} \neq 0; a_{22} = a_{21} = a_{12}$ $\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} a_{22} = a_{12} = a_{21} \neq 0$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$
D_7^7 : $e_1 \prec e_1 = e_1$ $e_2 \prec e_1 = -e_1$ $e_2 \prec e_2 = e_1 + e_2$ $e_2 \succ e_1 = e_1$ $e_2 \succ e_2 = -e_1$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{12} = a_{22} \neq 0$
$D_{8,\lambda}^7$: $e_1 \prec e_1 = \lambda e_2$ $e_1 \succ e_1 = e_1 - \lambda e_2$ $e_2 \succ e_2 = e_2$ $\lambda \neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} a_{12} = a_{22} \neq 0$

Associative algebra \mathcal{A}	Dendriform algebra structures	Solutions of D -equation
	$D_{9,\lambda}^7: e_1 \prec e_1 = \lambda e_2$ $e_2 \prec e_2 = e_2$ $e_1 \succ e_1 = e_1 - \lambda e_2$ $\lambda \neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$
	$D_{10,\lambda}^7: e_1 \prec e_2 = -e_2$ $e_1 \prec e_1 = e_1 - \lambda e_2$ $e_2 \prec e_1 = -e_2$ $e_1 \succ e_2 = e_2$ $e_2 \succ e_1 = e_2$ $e_1 \succ e_1 = \lambda e_2$ $e_2 \succ e_2 = e_2$ $\lambda \neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$ $\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0$
	$D_{11,\lambda}^7: e_1 \prec e_2 = -e_2$ $e_1 \prec e_1 = e_1 - \lambda e_2$ $e_2 \prec e_1 = -e_2$ $e_2 \prec e_2 = e_2$ $e_1 \succ e_2 = e_2$ $e_2 \succ e_1 = e_2$ $e_1 \succ e_1 = \lambda e_2$ $\lambda \neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix}$ $\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} a_{12} \neq 0$

4.2.1. *Symmetric solutions and Connes cocycles structures.* Using the Proposition 2.16, we obtain the results in Table 8, where D_i^j , with $i, j \in \mathbb{N}^*$, means the i -th dendriform class associated with the j -th class of associative algebra.

Table 8: Symmetric solutions and Connes cocycles structures of 2-dimensional associative algebras.

Dendriform algebra structures	Symmetric solutions	Connes cocycles structures over $\mathcal{A} \oplus \mathcal{A}^*$
$D_{1,\lambda}^1$	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_2; e_1 * e_2^* = \lambda e_1^*; e_2^* * e_1 = (1 - \lambda)e_1^*$
D_1^2	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_1 = e_2^*; e_1^* * e_2 = -a_{11}e_2$ $e_1 * e_1^* = -a_{11}e_1; e_2^* * e_1^* = -a_{11}e_2^*; e_1^* * e_1 = e_1^*$ $e_1^* * e_1^* = -a_{11}e_1^*$
	$a_{22}e_2 \otimes e_2$ $a_{22} \neq 0$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_1 = e_2^*;$ $e_1 * e_2^* = -a_{22}e_2; e_1^* * e_1 = e_1^*$
	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2 +$ $a_{12}e_2 \otimes e_1 +$ $a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1^* * e_1^* = -a_{11}e_1^*$ $e_1 * e_2^* = -a_{12}e_1; e_2^* * e_2^* = a_{22}e_1^*; e_1 * e_1^* = -a_{11}e_1 - a_{12}e_2$ $e_1^* * e_1 = a_{12}e_2 + e_1^*; e_2^* * e_1 = a_{22}e_2 + e_2^*;$ $a_{12}^2 = a_{11}a_{22}$
D_2^2	$a_{12}e_1 \otimes e_2 +$ $a_{12}e_2 \otimes e_1 +$ $a_{22}e_2 \otimes e_2$ $a_{12} \neq 0$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1^* * e_2^* = -a_{12}e_1^*; e_2^* * e_1^* = -a_{12}e_1^*$ $e_2^* * e_2^* = -a_{22}e_1^*; e_2^* * e_2 = -a_{12}e_2; e_1 * e_1^* = -a_{12}e_2 + e_1^*$ $e_1 * e_2^* = -a_{22}e_2 - a_{12}e_1; e_2^* * e_1 = a_{22}e_2 + e_2^*;$

Dendriform algebra structures	Symmetric solutions	Connes cocycles structures over $\mathcal{A} \oplus \mathcal{A}^*$
D_2^2	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_1 = e_2^*; e_1^* * e_1^* = -a_{11}e_1^*$ $e_1^* * e_1 = -a_{11}e_1^*; e_1^* * e_1^* = e_1^*$
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_1 * e_2^* = -a_{22}e_2$ $e_2^* * e_1 = a_{22}e_2 + e_2^*$
D_3^2	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_1^* = a_{12}e_1^*$
	$a_{12}e_1 \otimes e_2 +$	$e_1^* * e_2^* = -a_{12}e_1^*; e_1 * e_1^* = e_1^*; e_1^* * e_1^* = -a_{11}e_1^*$
	$a_{21}e_2 \otimes e_1$	$e_1^* * e_1 = -a_{11}e_1; e_1^* * e_2 = -a_{11}e_2; e_2^* * e_1 = -a_{12}e_1$
	$a_{22}e_2 \otimes e_2$	$e_2^* * e_2 = -a_{12}e_2; e_1 * e_2^* = -a_{22}e_2 - a_{12}e_1;$
	$a_{21} = a_{12}$	$e_2 * e_2^* = a_{12}e_2 + a_{11}e_1 + e_1^*; e_2^* * e_2^* = -a_{22}e_1^*$
D_4^2	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_1 * e_2^* = e_2^*$ $e_1^* * e_1 = -a_{11}e_1; e_1^* * e_2 = -a_{11}e_2; e_2^* * e_1 = e_2^*; e_1^* * e_1^* = -a_{11}e_1^*$ $e_1^* * e_2^* = -a_{11}e_2^*; e_2^* * e_1^* = -a_{11}e_2^*; e_2^* * e_2 = -a_{11}e_1 - e_1^*$
	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_2^* = -a_{12}e_2^*$
D_5^2	$a_{12}e_1 \otimes e_2 +$	$e_1^* * e_1^* = -a_{11}e_1^*; e_1 * e_1^* = e_1^*; e_1^* * e_1 = -a_{11}e_1$
	$a_{12}e_2 \otimes e_1$	$e_1^* * e_2 = -a_{11}e_2; e_2^* * e_2 = -a_{12}e_2$
	$a_{22}e_2 \otimes e_2$	$e_2^* * e_1 = (-a_{12} + a_{22})e_2 - e_1^* + e_2^*$
	$a_{21} = a_{12} \neq 0$	$e_1 * e_2^* = (a_{12} - a_{22})e_2 + e_1^* + (a_{11} - a_{12})e_1$ $e_2^* * e_1^* = (a_{11} - a_{12})e_1^* - a_{11}e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1 * e_1^* = e_1^*$ $e_1^* * e_1 = e_1^*; e_1 * e_2^* = -a_{22}e_2 + e_1^*$ $e_2^* * e_1 = a_{22}e_2 + e_2^* - e_1^*$
D_1^3	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1^* * e_1 = e_1^*; e_2^* * e_1 = -a_{22}e_2$
	$a_{22} \neq 0$	$e_1 * e_2^* = e_2^* + a_{22}e_2$
	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1^* * e_1 = e_1^*; e_1 * e_2^* = e_2^*$ $e_1 * e_1^* = -a_{11}e_1; e_2 * e_1^* = -a_{11}e_2$ $e_1^* * e_1^* = -a_{11}e_1^*; e_1^* * e_2^* = -a_{11}e_2^*$
	$a_{12}e_1 \otimes e_2 +$	
	$a_{21}e_2 \otimes e_1$	
D_2^3	$a_{12} = a_{21} \neq 0$	
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1 * e_1^* = e_1^*$ $e_2^* * e_1 = -a_{22}e_2; e_1 * e_2^* = e_2^* + a_{22}e_2$
D_3^3	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1 * e_1^* = e_1^*; e_1 * e_2^* = e_2^*$ $e_2 * e_1^* = -a_{11}e_2; e_1^* * e_1 = -a_{11}e_1$
	$a_{12}e_1 \otimes e_2 +$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1 * e_1^* = -a_{11}e_1; e_2 * e_1^* = -a_{11}e_2$
D_3^3	$a_{12}e_2 \otimes e_1 +$	$e_2 * e_2^* = -a_{12}e_2; e_1^* * e_1^* = -a_{11}e_1^*; e_2^* * e_1^* = a_{11}e_1^*$
	$a_{11}e_1 \otimes e_1 +$	$e_2^* * e_2^* = -a_{12}e_2^*; e_1 * e_2^* = a_{11}e_1 + (a_{12} + a_{22})e_2 + e_1^* + e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_1^* * e_1 = a_{11}e_2 + e_1^*; e_1^* * e_2^* = (-a_{11} - a_{12})e_1^* - a_{11}e_2^*$
	$a_{22} = a_{11}$	$e_2^* * e_1 = (-a_{11} - a_{12})e_1 + (-a_{12} - a_{22})e_2 - e_1^*$
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1 * e_1^* = e_1^*; e_2 * e_2^* = -e_1^*$ $e_2^* * e_2^* = a_{22}e_1^*; e_2^* * e_1 = -e_1^* + e_2^*; e_1 * e_2^* = a_{22}e_2 + e_1^* + e_2^*$
D_4^3	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_2^* * e_2^* = a_{22}e_1^*; e_2^* * e_1 = e_2^*$ $e_1^* * e_1 = e_1^*; e_2 * e_2^* = -e_1^*; e_1 * e_2^* = a_{22}e_2 + e_2^*$
	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_2 * e_1 = e_2; e_1^* * e_1^* = -a_{11}e_1^*; e_2^* * e_1^* = -a_{11}e_2^*$
D_1^4	$a_{12}e_1 \otimes e_2 +$	$e_2^* * e_2^* = (a_{22} - a_{12})e_1^* - 2a_{12}e_2^*; e_1^* * e_2^* = -a_{11}e_2^*$
	$a_{12}e_2 \otimes e_1 +$	$e_2 * e_1^* = -a_{11}e_2; e_1 * e_2^* = -a_{12}e_1 + (a_{12} + a_{22})e_2 + e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_2 * e_2^* = -a_{11}e_1 - 2a_{12}e_2 - e_1^*; e_1 * e_1^* = -a_{11}e_1$
	$a_{11}a_{22} = a_{12}^2; a_{11} \neq 0$	$e_2^* * e_1 = -a_{12}e_2 + e_2^*; e_1^* * e_1 = a_{12}e_1 + e_1^*$
	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2 * e_1 = e_2; e_2^* * e_1 = e_2^*$
D_1^4	$a_{22}e_2 \otimes e_2$	$e_1^* * e_1^* = -a_{11}e_1^*; e_1^* * e_2^* = -a_{11}e_2^*; e_1^* * e_2^* = -a_{11}e_2^*$ $e_1 * e_2^* = e_2^*; e_2^* * e_1^* = -a_{11}e_2^*; e_2 * e_1^* = -a_{11}e_2$ $e_1 * e_1^* = e_1^*; e_1^* * e_1 = -a_{11}e_1$

Dendriform algebra structures	Symmetric solutions	Connes cocycles structures over $\mathcal{A} \oplus \mathcal{A}^*$
D_2^4	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2^* * e_2 = a_{11}e_1 + e_1^*$ $e_2^* * e_1 = e_2^*; e_1 * e_1^* = -a_{11}e_1$ $e_1^* * e_1^* = -a_{11}e_1^*; e_2^* * e_1^* = -a_{11}e_2^*; e_2 * e_1 = e_2$ $e_2 * e_1^* = -a_{11}e_2; e_1^* * e_2 = -a_{11}e_2; e_1^* * e_1 = e_1^*$
	$a_{22}e_2 \otimes e_2 +$ $a_{12}e_1 \otimes e_2 +$ $a_{21}e_2 \otimes e_1$	$e_1 * e_1 = e_1; e_1 * e_1^* = -a_{12}e_2; e_2 * e_2^* = -a_{12}e_2$ $e_2 * e_1 = e_2; e_2^* * e_2 = e_1^*; e_1 * e_2^* = -a_{12}e_1 - a_{22}e_2$ $e_2^* * e_1 = e_2^* + a_{12}e_2; e_1^* * e_1 = e_1^* - a_{12}e_2$ $e_1 * e_2 = e_2; e_1^* * e_2^* = -a_{12}e_1^*; e_2^* * e_1^* = -2a_{12}e_1^*$ $e_2^* * e_2^* = (a_{12} - a_{22})e_1^* - a_{12}e_2^*$
D_3^4	$a_{11}e_1 \otimes e_1 +$ $a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_1^* * e_1^* = -a_{11}e_1^*; e_1^* * e_2^* = -a_{11}e_2^*$ $e_2 * e_1 = e_2; e_1^* * e_1 = e_1^*; e_2^* * e_1^* = -a_{11}e_2^*; e_1 * e_1^* = -a_{11}e_1$ $e_1 * e_2^* = e_2^*; e_2^* * e_1 = e_2^*; e_2 * e_1^* = -a_{11}e_2^*; e_1^* * e_2 = -a_{11}e_2$
D_4^4	$a_{11}e_1 \otimes e_1 +$ $a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_1 * e_2 = e_2; e_2 * e_1 = e_2; e_1 * e_1^* = e_1^*$ $e_2^* * e_2^* = -a_{22}e_1^*; e_1^* * e_2^* = -a_{11}e_2^*; e_1^* * e_1^* = -a_{11}e_1^*$ $e_2 * e_1^* = -a_{11}e_2; e_1^* * e_1 = -a_{11}e_1; e_1^* * e_2 = -a_{11}e_2$ $e_2^* * e_2^* = a_{11}e_1^* - a_{11}e_2^*; e_2^* * e_2^* = a_{11}e_1 - a_{22}e_2 + e_1^*$ $e_2^* * e_1 = -a_{11}e_1^* + e_2^* - e_1^*; e_2^* * e_2 = a_{11}e_1 + e_1^*$
D_1^5 $\beta = 0, 1$	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2 +$ $a_{12}e_{21} \otimes e_1 +$ $a_{22}e_2 \otimes e_2;$	$e_i * e_j = e_i^* * e_j = 0$ $e_i * e_j^* = e_i^* * e_j^* = 0$ $i, j = 1, 2; \beta = 0$
	$a_{11}e_1 \otimes e_1$	$e_1^* * e_2 = e_2^*; e_2 * e_1^* = -e_2^*; \beta = 1$
D_1^6	$a_{11}e_1 \otimes e_1 +$ $a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2^* * e_2^* = -a_{22}e_2^*$ $e_2 * e_2^* = -a_{22}e_2; e_2^* * e_2 = e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2^* * e_2 = e_2^*; e_2^* * e_2^* = -a_{22}e_2^*; e_1^* * e_2 = e_1^*$ $e_1^* * e_1^* = a_{22}e_1^*; e_1^* * e_1 = a_{22}e_1; e_2^* * e_1 = -a_{22}e_1$ $e_2^* * e_2^* = -a_{22}e_2; e_1 * e_1^* = -a_{22}e_2 - e_2^*$
D_2^6	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2 +$ $a_{12}e_2 \otimes e_1$	$e_2 * e_2 = e_2; e_2^* * e_2 = e_2^*; e_1^* * e_2 = e_1^*; e_1 * e_1^* = -e_2^*$ $e_2 * e_1^* = -a_{12}e_2; e_1^* * e_1 = a_{11}e_1; e_2^* * e_1 = a_{12}e_1$ $e_1^* * e_2^* = -a_{12}e_2^* + a_{12}e_1^*; e_1^* * e_1^* = a_{11}e_1^*$
	$a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2 * e_1^* = e_1^*; e_2 * e_2^* = e_2^*$ $e_2^* * e_1^* = -a_{22}e_1^*; e_1 * e_1^* = -e_2^* - a_{22}e_2$
	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2 +$ $a_{12}e_2 \otimes e_1$	$e_2 * e_2 = e_2; e_2^* * e_2^* = a_{12}e_2^*; e_2 * e_1^* = -a_{12}e_2$ $e_2 * e_1^* = e_1^* + a_{11}e_1; e_2 * e_2^* = e_2^* + a_{12}e_2$ $e_1 * e_1^* = -e_2^* - a_{12}e_1; e_1^* * e_1^* = a_{11}e_2^* - a_{12}e_1^*$
D_3^6	$a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2 * e_1^* = e_1^*; e_2 * e_2^* = e_2^*$ $e_2^* * e_1^* = -a_{22}e_1^*; e_1 * e_1^* = -e_2^* - a_{22}e_2$
	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2 +$ $a_{12}e_2 \otimes e_1$	$e_2 * e_2 = e_2; e_2^* * e_2^* = a_{12}e_2^*; e_2 * e_1^* = -a_{12}e_2$ $e_2 * e_1^* = e_1^* + a_{11}e_1; e_2 * e_2^* = e_2^* + a_{12}e_2$ $e_1 * e_1^* = -e_2^* - a_{12}e_1; e_1^* * e_1^* = a_{11}e_2^* - a_{12}e_1^*$
	$a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2^* * e_1 = e_1^*; e_2^* * e_2^* = -a_{22}e_2^*$ $e_2 * e_2^* = -a_{22}e_2; e_1 * e_2^* = -e_1^*; e_2^* * e_2 = e_2^*$
D_4^6	$a_{22}e_2 \otimes e_2$	$e_2 * e_2 = e_2; e_2^* * e_1 = e_1^*; e_2^* * e_2^* = -a_{22}e_2^*$ $e_2 * e_2^* = -a_{22}e_2; e_1 * e_2^* = -e_1^*; e_2^* * e_2 = e_2^*$
	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_2^* * e_1^* = -a_{12}e_2^*; e_2^* * e_2^* = -a_{22}e_2^*$
	$a_{12}e_1 \otimes e_2 +$	$e_1 * e_2^* = -a_{11}e_1; e_1 * e_2^* = -a_{12}e_1; e_2 * e_1^* = -a_{12}e_2^*$
	$a_{21}e_2 \otimes e_1$	$e_2 * e_2^* = -a_{22}e_2; e_1^* * e_2 = -a_{12}e_2; e_2^* * e_1 = -a_{12}e_1$
	$a_{22}e_2 \otimes e_2$ $a_{12} = a_{21}$	$e_1^* * e_1^* = -a_{11}e_1^*; e_1^* * e_2^* = -a_{12}e_1^*$ $e_1^* * e_1 = a_{12}e_2 + e_1^*; e_2^* * e_2 = a_{12}e_1 + e_2^*$
D_2^7	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_1^* = -a_{11}e_1; e_1 * e_2^* = -a_{12}e_1$
	$a_{12}e_1 \otimes e_2 +$	$e_1^* * e_2 = -a_{12}e_2; e_2^* * e_2 = -a_{22}e_2; e_1^* * e_1^* = (-a_{11} - a_{12})e_1^*$
	$a_{21}e_2 \otimes e_1$	$e_2^* * e_1^* = -a_{22}e_1^*; e_2^* * e_2^* = -a_{22}e_2^*; e_1^* * e_2^* = -a_{12}e_2^* - a_{12}e_1^*$
	$a_{22}e_2 \otimes e_2$	$e_2 * e_1^* = -a_{12}e_2 + e_1^*; e_2 * e_2^* = a_{12}e_1 + e_2^*$
	$a_{12} = a_{21}$	$e_1^* * e_1 = -a_{12}e_1 + (a_{12} - a_{22})e_2 + e_1^* - e_2^*$
D_3^7	$a_{11}e_1 \otimes e_1 +$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1^* * e_1^* = -a_{11}e_1^*; e_2^* * e_2^* = -a_{22}e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1^* = -a_{11}e_1; e_2 * e_2^* = e_2^*; e_1^* * e_1 = e_1^*; e_2^* * e_2 = -a_{22}e_2$

Dendriform algebra structures	Symmetric solutions	Connes cocycles structures over $\mathcal{A} \oplus \mathcal{A}^*$
D_4^7	$a_{11}e_1 \otimes e_1 +$ $a_{12}e_1 \otimes e_2$ $a_{21}e_2 \otimes e_1$ $a_{21} = a_{12}$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_2^* * e_1^* = -a_{11}e_2^*; e_1^* * e_2^* = -a_{12}e_2^*$ $e_1 * e_1^* = -a_{11}e_1; e_1 * e_2^* = -a_{12}e_1; e_2 * e_1^* = -a_{12}e_1^*$ $e_1^* * e_2 = -a_{12}e_2; e_2^* * e_1 = e_2^*; e_1^* * e_1^* = -a_{11}e_1^*$ $e_2^* * e_2^* = -a_{12}e_2^*; e_1^* * e_1 = a_{12}e_2 + e_1^*$ $e_2 * e_2^* = (-a_{11} + a_{12})e_1 - a_{12}e_2 - e_1^* + e_2^*$
$D_{5,\lambda}^7$	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_1^* * e_1 = -a_{11}e_1$ $e_1 * e_2^* = -\lambda e_2^*; e_1^* * e_1^* = -a_{11}e_1^*; e_1^* * e_2^* = a_{11}\lambda e_2^*$ $e_2^* * e_2 = a_{11}\lambda e_1 + \lambda e_1^* + e_2^*$
	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_1 * e_2^* = -a_{22}\lambda e_2$ $e_2 * e_2^* = -a_{22}e_2; e_1 * e_2^* = -\lambda e_2^*; e_2^* * e_2 = \lambda e_1^* + e_2^*$ $e_2^* * e_2^* = -\lambda a_{22}e_1^* - a_{22}e_2^*$
	$a_{11}e_1 \otimes e_1$ $a_{12}e_1 \otimes e_2$ $a_{21}e_2 \otimes e_1$ $a_{22}e_2 \otimes e_2$ $a_{21} = a_{12}$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_1^* = e_1^*; e_2 * e_1^* = -a_{12}e_2$ $e_2 * e_2^* = -a_{22}e_2; e_1^* * e_1 = -a_{11}e_1; e_1^* * e_2 = -a_{12}e_2$ $e_2^* * e_1 = -a_{12}e_1; e_1^* * e_2^* = -a_{12}e_2^*; e_1^* * e_1^* = -a_{11}e_1^*$ $e_2^* * e_1^* = -a_{12}e_2^*; e_2^* * e_2 = -a_{12}e_2 - e_1^* + e_2^*$ $e_2^* * e_2^* = -a_{22}e_1^* + (-a_{22} - a_{12})e_2^*$
D_6^7	$a_{11}e_1 \otimes e_1$ $a_{12}e_1 \otimes e_2$ $a_{21}e_2 \otimes e_1$ $a_{22}e_2 \otimes e_2$ $a_{21} = a_{12} = a_{22}$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_1^* = -a_{12}e_1 + e_1^* - e_2^*$ $e_1 * e_2^* = -a_{12}e_1; e_2 * e_1^* = -a_{12}e_2; e_2 * e_2^* = -a_{22}e_2$ $e_1^* * e_2^* = -a_{12}e_1^*; e_2^* * e_1 = -a_{12}e_1; e_1^* * e_1 = -a_{11}e_1$ $e_2^* * e_2^* = (-a_{22} - a_{12})e_2^*; e_1^* * e_2 = a_{11}e_1 + a_{12}e_2 + e_1^*$ $e_2^* * e_1^* = -a_{12}e_1^* + a_{12}e_2^*; e_2^* * e_2 = a_{12}e_1 - a_{12}e_2 + e_2^*$ $e_1^* * e_1^* = (-a_{11} - a_{12})e_1^* + a_{11}e_2^*$
D_7^7	$a_{11}e_1 \otimes e_1$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_2 * e_1^* = e_2^*; e_2 * e_2^* = e_2^*$ $e_1^* * e_1 = -a_{11}e_1; e_1^* * e_2 = a_{11}e_1 + e_1^* - e_2^*$ $e_1 * e_1^* = e_1^* - e_2^*; e_1^* * e_1^* = -a_{11}e_1^* + a_{11}e_2^*$
$D_{8,\lambda}^7$	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_2^* = \lambda e_1^*; e_2 * e_2^* = -a_{22}e_2$ $e_1^* * e_1 = e_1^*; e_2^* * e_1 = -\lambda e_1^*; e_2^* * e_2 = e_2^*; e_2^* * e_2^* = -a_{22}e_2^*$
$D_{9,\lambda}^7$	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_2^* = \lambda e_1^*; e_2^* * e_2 = -a_{22}e_2$ $e_2 * e_2^* = e_2^*; e_1^* * e_1 = e_1^*; e_2^* * e_1 = -\lambda e_1^*; e_2^* * e_2^* = -a_{22}e_2^*$
$D_{10,\lambda}^7$	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_2^* = -a_{22}e_2 - \lambda e_1^* - e_2^*$ $e_1 * e_1^* = e_1^*; e_2^* * e_2^* = -a_{22}e_2^*; e_2 * e_2^* = -a_{22}e_2 - e_1^*$ $e_2^* * e_2 = e_1^* + e_2^*; e_2^* * e_1 = \lambda e_1^* + e_2^*$
$D_{11,\lambda}^7$	$a_{22}e_2 \otimes e_2$	$e_1 * e_1 = e_1; e_2 * e_2 = e_2; e_1 * e_2^* = -a_{22}e_2 - \lambda e_1^* - e_2^*$ $e_1 * e_1^* = e_1^*; e_2^* * e_2^* = -a_{22}e_2^*; e_2 * e_2^* = e_2^* - e_1^*$ $e_2^* * e_2 = -a_{22}e_2 + e_1^*; e_2^* * e_1 = a_{22}e_2 + \lambda e_1^* + e_2^*$

5. Concluding remarks

In this work, we gave an overview of the main concepts, definitions and known fundamental results related to the notions of Frobenius algebras, bialgebras, and Connes cocycles. We classified the solutions of the associative Yang-Baxter equation on complex associative algebras in dimensions 1 and 2. In dimension 1, we obtained 2 classes against 7 in dimension 2. The skew-symmetric solutions enabled us to carry out double constructions of the Frobenius algebras of these associative algebras. Finally, we obtained all compatible dendriform algebras in dimensions 1 and 2, gave a classification of solutions of D -equations and the skew-symmetric Connes cocycles associated with symmetric solutions.

Aknowledgement

MNH is grateful to Prof C. Bai from Chern Institute of Mathematics, China, for useful discussions, inputs and provided references. This work is partially supported by the Abdus Salam

International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the Office of External Activities (OEA) - Prj-15. The ICMPA is also in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

References

- [1] M. Aguiar, Infinitesimal Hopf algebras. In New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math. Amer. Math. Soc., Providence, RI, **267** (2000), pp. 1-29.
- [2] M. Aguiar, Pre-Poisson algebras. Lett. Math. Phys. **54** (2000), pp. 263-277.
- [3] M. Aguiar, On the associative analog of Lie bialgebras. J. Algebra **244** (2001), pp. 492-532.
- [4] H. An and C. Bai, From Rota-Baxter algebras to pre-Lie algebras. J. Phys. A: Math. Theor. **41** (2008)015201(19pp).
- [5] C. Bai, Double constructions of Frobenius algebras, Connes cocycle and their duality. J. Noncommut. Geom. **4** (2010), pp. 475 - 530.
- [6] M. Bordemann, Nondegenerate invariant bilinear forms on non associative algebras. Acta Math. Univ. Comenian. (N.S.) **66** (1997), pp 151-201.
- [7] M. Bordemann, T. Filk, C. Nowak, Algebraic classification of actions invariant under generalized flip moves of 2-dimensional graph. J. Math. Phys. **35** (1994), pp. 4964-4988.
- [8] R. Brauer, C. Nesbitt, On the regular representations of algebras. Proc. Nat. Acad. Sci. U.S.A. **66** (1937), pp. 236-240.
- [9] D. Burde, Simple left-symmetric algebras with solvable Lie algebra. Manuscripta Math. 95, **3** (1998), pp. 397-411.
- [10] F. Chapoton, Un théorème de Cartier - Milnor - Moore - Quillen pour les bigèbres dendriformes et les algèbres braces. J. Pure and Appl. Alg. **168** (2002), pp. 1 - 18.
- [11] A. Connes Non-commutative differential geometry. Inst. Hautes Etudes Sci. Publ. Math. **62** (1985), pp. 257-360.
- [12] K. Ebrahimi-Fard, D. Manchon, F. Patras, New identities in dendriform algebras. J. Algebra **320** (2008), pp. 167-194.
- [13] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés II. Bull. Sci. Math. **126** (2002), pp. 249-288.
- [14] A. Frabetti, Dialgebra homology of associative algebras. C. R. Acad. Sci. Paris **325** (1997), pp. 135-140.
- [15] A. Frabetti, Leibniz homology of dialgebras of matrices. J. Pure. Appl. Alg. **129** (1998), pp. 123-141.
- [16] R. Holtkamp, Comparison of Hopf algebras on trees. Arch. Math. (Basel). **80** (2003), pp. 368-383.
- [17] R. Holtkamp, On Hopf algebra structures over operad. Adv. Math. **207** (2006), pp. 544-565.
- [18] S. A. Joni, G. C. Rota, Coalgebras and bialgebras in combinatorics. Stud. Appl. Math. **61** (1979), pp. 93-139.
- [19] J. Kock, Frobenius algebras and 2D topological quantum field theories. London Mathematical Society Student Texts, 59. Cambridge University Press, Cambridge, 2004.
- [20] J.-L. Loday, Dialgebras, in Dialgebras and related operads. Lecture Notes in Math. **1763** (2002), pp. 7-66.
- [21] J.-L. Loday, Arithmetree. J. Algebra **258** (2002), pp. 275-309.
- [22] J.-L. Loday, Scindement d'associativité et algèbres de Hopf. Proceedings of the Conference in honor of Jean Leray, Nantes (2002), Séminaire et Congrès(SMF) **9** (2004), pp. 155-172.
- [23] J.-L. Loday, M. Ronco, Hopf algebra of the planar binary trees. Adv. Math. **139** (1998), pp. 293-309.
- [24] J.-L. Loday, M. Ronco, Algèbre de Hopf colibres. C.R. Acad. Sci. Paris **337** (2003), pp. 153-158.
- [25] M. Ronco, Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras. J. Algebra **254** (2002), pp. 151-172.
- [26] R. D. Schafer, An introduction to nonassociative algebras. Corrected reprint of the 1966 original, Dover Publications, New York (1995).
- [27] K. Yamagata, Frobenius algebras. In Handbook of algebra, Vol. 1, North-Holland, Amsterdam (1996), pp. 841-887.
- [28] L. Zhang, The classification of 2-dimensional dendriform algebras. Thesis for Bachloar Degree, Nankai University, (2008).
- [29] V. N. Zhelyabin, Jordan bialgebras and their connection with Lie bialgebras. Algebra i Logika **36** (1997), pp. 3-25; English transl. Algebra Logic **36** (1997), pp. 1-16.

(*) UNIVERSITY OF ABOMEY-CALAVI, INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, ICMPA-UNESCO CHAIR, 072 BP 50, COTONOU, REP. OF BENIN

E-mail address: `norbert.hounkonnou@cipma.uac.bj`, with copy to `hounkonnou@yahoo.fr`

(†) UNIVERSITY OF ABOMEY-CALAVI, INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, ICMPA-UNESCO CHAIR, 072 BP 50, COTONOU, REP. OF BENIN

E-mail address: `houndedjid@gmail.com`